STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

2022

HURLSTONE AGRICULTURAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- NESA approved calculators may be used
- A reference sheet is provided in the Section I booklet
- For questions in Section II, show all relevant mathematical reasoning and/or calculations
- This examination paper is not to be removed from the examination centre

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10. The multiple-choice answer sheet has been provided
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 11)

- Attempt Questions 11 – 16, write your solutions in the spaces provided.
- There are 6 separate question/answer booklets. Extra working pages are available if required.
- Allow about 2 hours and 45 minutes for this section.

Disclaimer: Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2022 HSC Mathematics Advanced Examination.

Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 What is the distance from the origin to the point $\begin{pmatrix} -2 \\ 3\sqrt{3} \\ 4 \end{pmatrix}$?

A. $2+3\sqrt{3}$

B. $\sqrt{29}$

C. $\sqrt{47}$

D. $24\sqrt{3}$

2 Let the point P on an Argand diagram represent the complex number z. After being multiplied by another complex number, ω, P is rotated 90° clockwise and $|z|$ is enlarged by a factor of 3. Which of the following is the value of ω?

A. $3i$

B. $-3i$

C. e^{-3i}

D. $3e^{\frac{\pi}{2}i}$

3 Consider the statement:

"$\exists x \in R,\ \ln x = 1$ and $x > 2$."

Which of the following is the negation of this statement?

A. $\exists x \in R,\ \ln x \neq 1$ or $x \leq 2$

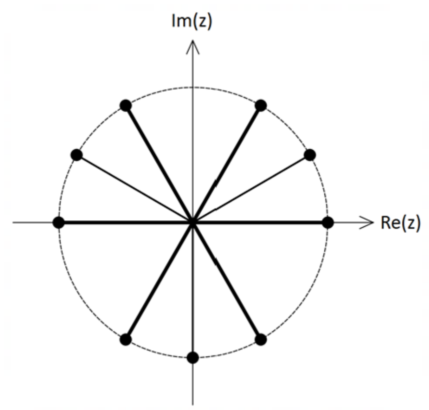
B. $\exists x \in R,\ \ln x \neq 1$ and $x \leq 2$

C. $\forall x \in R,\ \ln x \neq 1$ or $x \leq 2$

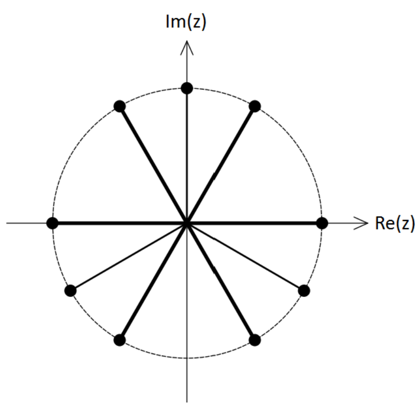
D. $\forall x \in R,\ \ln x \neq 1$ and $x \leq 2$

4 Which Argand diagram shows the solutions of $(z^3 - i)(z^6 - 1) = 0$?

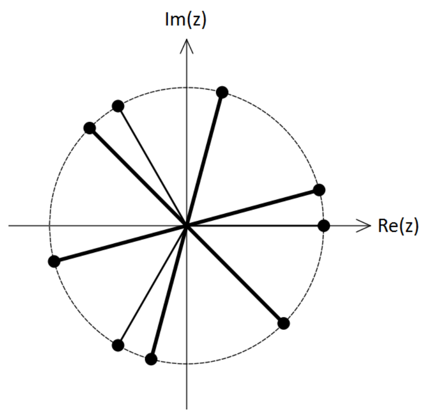
A.



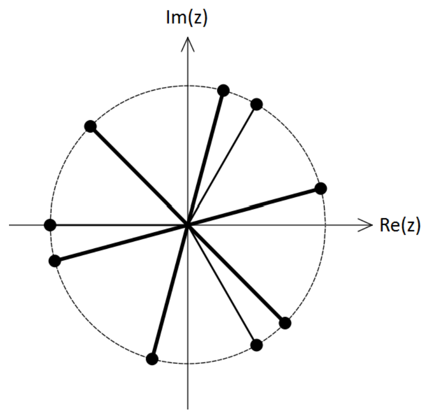
B.



C.



D.



5 In the complex plane, a circle C has diameter AB, where the points A and B represent the complex numbers $1-5i$ and $3-i$ respectively.

What is the equation of C?

A. $|z-2+3i| = 2\sqrt{5}$

B. $|z-2+3i| = \sqrt{5}$

C. $|z+2-3i| = 2\sqrt{5}$

D. $|z+2-3i| = \sqrt{5}$

6 The integral $\int \left(\frac{1}{x\sqrt{4+x^2}} \right) dx$ can be most efficiently solved with which substitution?

A. $x = \tan\theta$

B. $x = \sec\theta$

C. $x = 2\sec\theta$

D. $x = 2\tan\theta$

7 Two vectors are such that $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$.

What is the correct evaluation of $\left|\underset{\sim}{u}\right| \times \underset{\sim}{v} \cdot \underset{\sim}{v}$?

A. 84

B. 504

C. $6\sqrt{14}$

D. $36\sqrt{14}$

8 A particle moves in a straight line with simple harmonic about a centre *O*. The period of motion is π seconds. When the particle is 0.50 metres from *O*, its speed is 2.40 m/s.

What is the particle's maximum speed?

A. 2.40 m/s

B. 2.60 m/s

C. 3.38 m/s

D. 5.20 m/s

9 Which expression is equal to $\int \tan^{-1} x \, dx$?

A. $x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

B. $x^2\tan^{-1}x - \int \frac{x^2}{1+x^2}\,dx$

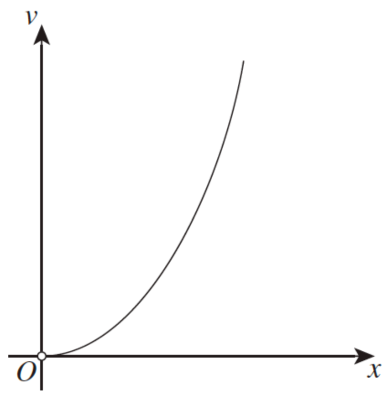
C. $\frac{1}{2}\left(\tan^{-1}x\right)^2$

D. $\frac{x}{2}\left(\tan^{-1}x\right)^2 - \int \frac{x}{\tan x}\,dx$

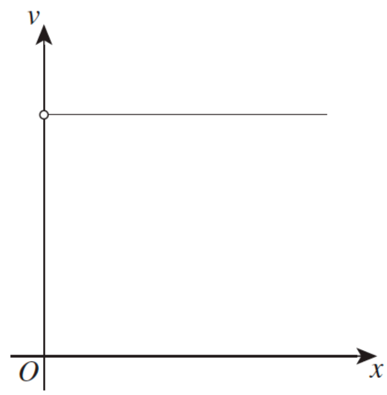
10 A particle moves in a straight line such that its acceleration a is given by $a = vx$, where v is the particle's velocity, x is the particle's position and v, $x > 0$.

Which of the following diagrams best shows the relationship between v and x?

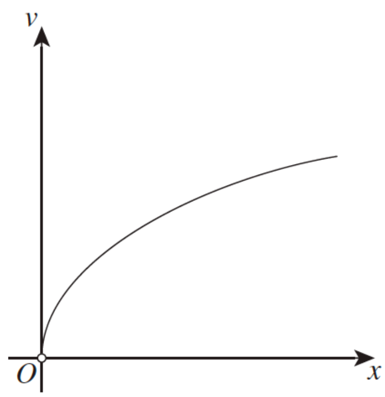
A.



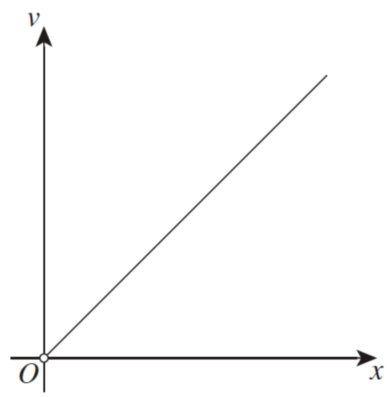
B.



C.



D.



END OF SECTION I

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer the questions in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

MARKS

(a) The complex number $u = 3 - i$ is represented on an Argand diagram by the point A. Points B and C, respectively, represent the complex numbers $\overline{u}$ and $\overline{u} - u$, and O represents the origin.

(i) Sketch points A, B and C on an Argand diagram. **2**

(ii) What type of quadrilateral is $OABC$? **1**

(iii) Calculate $\dfrac{\overline{u}}{u}$, expressing your answer in the form $x + iy$ **1**

where x and y are real.

(iv) Deduce that $\tan^{-1}\left(\dfrac{3}{4}\right) = 2\tan^{-1}\left(\dfrac{1}{3}\right)$. **2**

(b) Find the primitive of $\dfrac{4x-1}{x^2+2x+6}$. **3**

(c) If $x = 1 - 2i$ is a root of $x^3 + ax^2 - x + 15$, find a if a is a real number. **2**

(d) When an object moves through water, it experiences resistance proportional to its speed.

(i) If an object of unit mass moving at 10 m/s in water experiences a resistance of 40 Newtons,
write an expression for the acceleration due to resistance, in terms of v. **1**

(ii) A metal ball of unit mass is released from rest in water and immediately sinks. Given that $g = 10$ m/s^2, and after t seconds the velocity of the ball is v m/s, find the velocity of the metal ball as a function of t. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet.

MARKS

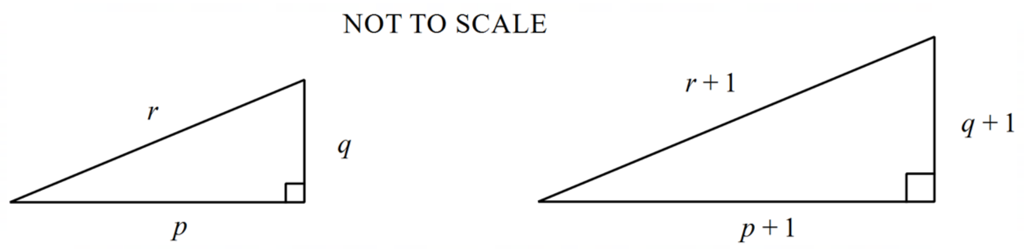
(a) Relative to a fixed origin O, the point A has position vector $2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}$, the point B has position vector $4\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$, and the point C has position vector $a\underset{\sim}{i}+5\underset{\sim}{j}-2\underset{\sim}{k}$, where a is a constant and $a<0$.

D is the point such that $\overrightarrow{AB}=\overrightarrow{BD}$.

(i) Find the position vector of D. **2**

(ii) If $\left|\overrightarrow{AC}\right|=4$, find the value of a. **3**

(b) Consider the figure below, which shows two right-angled triangles.



Prove by contradiction that p, q and r cannot all be integers. **3**

(c) A particle moves in a straight line. At time t seconds, its displacement from a fixed origin is x metres and its velocity is v m/s. The acceleration of the particle is given by $\ddot{x}=x+3$.
At $t=0$, the particle is at the origin and moving with velocity 3 m/s.

(i) Show that $v=x+3$ **2**

(ii) Find an expression for x, the displacement of the particle, in terms of t. **2**

d) A sphere is represented by the equation

$$x^2+2x+y^2-5y+z^2-4z+k=0.$$

(i) Determine the centre, $\underset{\sim}{c}$, and radius of the sphere. **2**

(ii) It is given that the $x-z$ plane is tangential to the sphere.
Determine the value of k. **1**

Question 13 (15 marks) Use the Question 13 Writing Booklet.

MARKS

(a) Consider the following statement, where x is an integer.

'If $x^2 - 6x + 5$ is even, then x is odd.'

(i) Write down the contrapositive of the statement. **1**

(ii) Prove the statement by proving the contrapositive. **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{2}{1+\sin x+\cos x}\,dx$. **3**

(c) Consider $I_n = \int x^m \ln^n x\,dx$

(i) Show that $I_n = \frac{x^{m+1}}{m+1}\ln^n x - \frac{n}{m+1}I_{n-1}$ **2**

(ii) Hence, evaluate $I_n = \int_1^2 x^3 \ln^4 x\,dx$ **3**

(d) Consider the complex number $z = i$.

(i) Explain why z can be written as $e^{i\frac{\pi}{2}}$ **1**

(ii) Hence prove that $i^i \approx 0.208$. **2**

(iii) By considering another value of arg(i) find a value of i^i that is greater than 1. **1**

Question 14 (15 marks) Use the Question 14 Writing Booklet.

MARKS

(a) A particle P of mass m kg is released from rest and falls under its own weight. The particle is subject to air resistance of magnitude kmv^2N, where v ms^{-1} is the speed of the particle after it has travelled x m and k is a positive constant.

(i) Show that $v^2 = \left(1 - e^{-2kx}\right)\dfrac{g}{k}$. **3**

(ii) Hence determine the terminal velocity, V_T, of the particle. **1**

(iii) Show that the particle attains a speed of $\dfrac{1}{2}V_T$ after it has fallen a distance of $\dfrac{1}{2k}\ln\left(\dfrac{4}{3}\right)$ metres. **2**

(b) A particle is moving on a straight line. Its velocity v is given by $v^2 = 4\left(2x - x^2\right)$ where x is its displacement from a point O on the line.

(i) Show that its acceleration is given by $\ddot{x} = -4\left(x - 1\right)$. **2**

(ii) Explain why this particle moves in simple harmonic motion. **1**

(iii) Find the maximum speed of the particle. **1**

(c) Consider points $A(1, 3, 1)$, $B(2, 5, -1)$ and $C(7, 5, -1)$.
Line L passes through point B and bisects $\angle ABC$.

Find the vector equation of line L. **2**

(d) Find the roots of the complex equation $\omega^4 = \left(\omega - 2\right)^4$ in the form $a + bi$ where $a, b \in R$. **3**

Question 15 (15 marks) Use the Question 15 Writing Booklet.

MARKS

(a) With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations:

$l_1 : r = 9\underset{\sim}{i} + 13\underset{\sim}{j} - 3\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$ and $l_2 : r = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + \mu\left(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

where λ and μ are scalar parameters.

(i) Given that l_1 and l_2 meet, find the position vector of their point of intersection. **3**

(ii) Find the acute angle between l_1 and l_2, correct to the nearest tenth of a degree. **2**

(iii) Given that the point A has position vector $4\underset{\sim}{i} + 16\underset{\sim}{j} - 3\underset{\sim}{k}$, and that the point $P(x_1, y_1, z_1)$ lies on l_1 such that AP is perpendicular to l_1, find the exact coordinates of P. **3**

(b) (i) Using de Moivre's theorem, show that **3**

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

(ii) Hence show that $\tan^2\left(\frac{\pi}{5}\right)$ and $\tan^2\left(\frac{2\pi}{5}\right)$ are the roots of the equation $x^2 - 10x + 5 = 0$. **2**

(c) For a complex number z, shade the region of the Argand Plane in which **2**

$$|z| < 3 \text{ and } -2 < \operatorname{Im}(z) < 2.$$

Question 16 (15 marks) Use the Question 16 Writing Booklet.

MARKS

(a) Find $\int \frac{\sqrt{x}}{4+x}\,dx$. **3**

(b) In a sequence of numbers, $T_1 = 7$ and $T_n = 2T_{n-1} - 1$ for $n \geq 2$.

Use mathematical induction to prove that $T_n = 3 \times 2^n + 1$ for $n \geq 1$. **3**

(c) Recall that $x + \frac{1}{x} \geq 2$ for any real number $x > 0$. **(DO NOT PROVE THIS)**

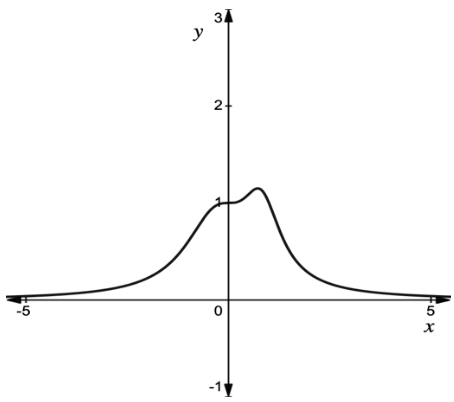
(i) Prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

for all real numbers $a > 0, b > 0, c > 0$. **2**

(ii) Hence prove that $\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right) \geq \frac{3}{2}$

for all real numbers $a > 0, b > 0, c > 0$. **3**

(d) Consider the graph of the function $f(x) = \frac{x^3+1}{x^5+1}$.



Using the substitution $u = \frac{1}{x}$, prove that the area under $f(x)$ between $x = 0$ and $x = 1$ is half the area under $f(x)$ for $x \geq 0$. **4**

End of paper

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2022 Trial Higher School Certificate Examination

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

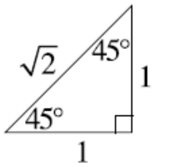
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$



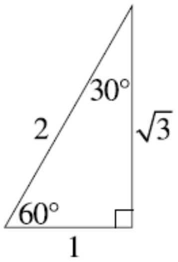
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

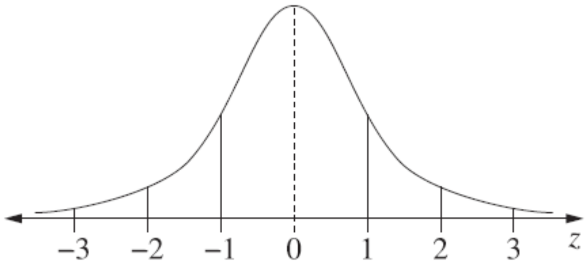
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\left[f(x_1)+\cdots+f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

HURLSTONE AGRICULTURAL HIGH SCHOOL

2022 Trial Higher School Certificate Examination
Mathematics Extension 2

Name ______________________ Teacher ________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ⊗ B ⊗ correct C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Year 12	Mathematics Extension 2	Ass Task 4 2022 HSC
MULTIPLE CHOICE	Solutions and Marking Guidelines	

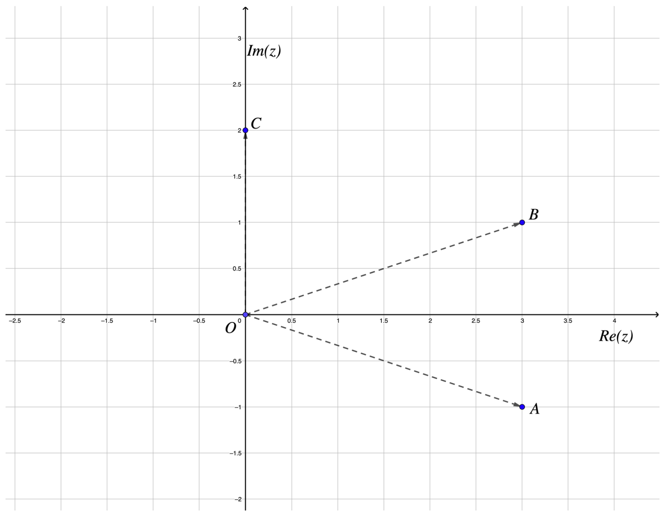
Part / Outcome	Solutions	ANSWER
1	distance $= \sqrt{(-2)^2 + \left(3\sqrt{3}\right)^2 + 4^2} = \sqrt{47}$	C
2	**B** is correct. Multiplication by $-i$ will cause a rotation of 90° clockwise, and multiplication by 3 will cause an enlargement factor of 3. A is incorrect. This option would cause a rotation of 90° anticlockwise. C and D are incorrect. The correct solution in Euler form is $\omega = -3e^{\frac{\pi}{2}i}$.	B
3	**C** is correct. In a negation statement, the phrases 'there exist' and 'for all' must be interchanged, 'and' and 'or' must be interchanged, '=' must change to '≠', and '>' must change to '≤'. A, B and D are incorrect. These statements do not show an accurate negation of the statement.	C
4	$z^6 = 1 \rightarrow z = 1 + 0i$ is one solution the 5 other solutions found by rotating by $\frac{\pi}{3}$. $\left(\right.$ ie 6 solutions evenly spaced around circle$\left.\right)$ $z^3 = i = e^{i\pi}$ $z = \left(e^{i\pi}\right)^{\frac{1}{3}} \rightarrow z = e^{\frac{\pi}{3}i}$ is one solution the 2 other solutions found by rotating by $\frac{2\pi}{3}$. $\left(\right.$ ie 3 solutions evenly spaced around circle$\left.\right)$	A
5	The centre of C is the midpoint of AB. ie $2 - 3i$. $AB = \sqrt{(3-2)^2 + (-1+5)^2} = 2\sqrt{5}$ so radius is $\sqrt{5}$. Equation of circle is $\left\lvert z - (2 - 3i)\right\rvert = \sqrt{5}$	B
6	$x = 2\tan\theta$ works best. That's it ☺	D

	$x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta d\theta$ so $\int \frac{1}{x\sqrt{4+x^2}}dx = \int \frac{2\sec^2\theta d\theta}{2\tan\theta\sqrt{4+4\tan^2\theta}}$ $= \int \frac{\sec^2\theta d\theta}{\tan\theta \times 2\sqrt{1+1\tan^2\theta}} = \frac{1}{2}\int \frac{\sec^2\theta d\theta}{\tan\theta\sqrt{\sec^2\theta}}$ $= \frac{1}{2}\int \frac{\sec^2\theta d\theta}{\tan\theta\sec\theta}$ etc... the main thing is, the binomial under the $\sqrt{\ }$ has been eliminated	
7	$\lvert \underset{\sim}{u} \rvert \times \underset{\sim}{v}\cdot\underset{\sim}{v} = \sqrt{2^2+1^2+3^2}\times\left(4^2+2^2+4^2\right)$ $= \sqrt{14}\times 36$	D
8	$T = \frac{2\pi}{n} = \pi \quad \Rightarrow \quad n = 2$ Using $v^2 = n^2\left(a^2 - x^2\right)$ with $v = 2.40, n = 2, x = 0.50$ $2.40^2 = 2^2\left(a^2 - 0.50^2\right)$ $a = 1.3$ m $v_{\max} = \sqrt{2^2\left(1.3^2 - 0^2\right)}$ ie when $x = 0$ (centre of motion) $= 2.60$ m/s	B
9	$u = \tan^{-1}x$ $\qquad v' = 1$ $u' = \frac{1}{1+x^2}$ $\qquad v = x\,dx$ $\int uv'dx = uv - \int u'v\,dx$ $\int \tan^{-1}dx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$	A
10	$a = vx$ ie $\quad v\frac{dv}{dx} = vx$ $\frac{dv}{dx} = x$ $\int dv = \int x\,dx$ $v = \frac{x^2}{2} + C$, which is a parabola $\therefore$ must be A	A

Year 12	Mathematics Extension 2 2022	TASK 4 (TRIAL)
Question No. 11	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-5 applies techniques of integration to structured and unstructured problems
MEX12-6 uses mechanics to model and solve practical problem

Part / Outcome	Solutions	Marking Guidelines
(a)(i) *MEX12-4*		**2 marks** – Correct solution **1 mark** – Substantially correct
(a)(ii) *MEX12-4*	$OABC$ is a parallelogram	**1 mark** – Correct solution
(a)(iii) *MEX12-4*	$\frac{\overline{u}}{u}=\frac{3+i}{3-i}$ $=\frac{3+i}{3-i}\times\frac{3+i}{3+i}$ $=\frac{(3+i)^2}{3^2+1^2}$ $=\frac{9+6i+i^2}{10}=\frac{4}{5}+\frac{3}{5}i$	**1 mark** – Correct solution
(a)(iv) *MEX12-4*	$\arg\left(\frac{\overline{u}}{u}\right)=\arg\overline{u}-\arg u$ $\arg\left(\frac{4}{5}+\frac{3}{5}i\right)=\arg(3+i)-\arg(3-i)$ $\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)=\tan^{-1}\left(\frac{1}{3}\right)-\tan^{-1}\left(\frac{-1}{3}\right)$ $=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{3}\right)$ $\tan^{-1}\left(\frac{3}{4}\right)=2\tan^{-1}\left(\frac{1}{3}\right)$ or, using $2\arg(\overline{u})=\arg(\overline{u})^2$	**2 marks** – Correct solution **1 mark** – Substantially correct **0 marks** – using a calculator NB: ‘deduce’… requires you to use info found in previous parts

	Question 11 continued…	
(b) *MEX12-5*	$$\int \frac{4x-1}{x^2+2x+6}dx = \int \frac{4x+4-5}{x^2+2x+6}dx$$ $$= 2\int \frac{2x+2}{x^2+2x+6}dx - 5\int \frac{1}{x^2+2x+6}dx$$ $$= 2\ln\left|x^2+2x+6\right| - 5\int \frac{1}{(x+1)^2+5}dx$$ $$= 2\ln\left|x^2+2x+6\right| - \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$$	**3 marks** – Correct solution **2 marks** – Substantially correct • Separates the expression into two fractions *AND* • Derives $2\ln\left\|x^2+2x+6\right\|$ **1 mark** – Partial progress towards correct solution • Separates the expression into two fractions *OR* • Derives $2\ln\left\|x^2+2x+6\right\|$ *OR* • Equivalent merit
(c) *MEX12-4*	Coefficients are real so if $1-2i$ is a root, then $1+2i$ is also a root. Product of roots $\quad \alpha\beta\delta = -\frac{d}{a}$ $$(1+2i)(1-2i)\alpha = -15$$ $$5\alpha = -15$$ $$\alpha = -3$$ Sum of roots $\quad \alpha+\beta+\delta = -\frac{b}{a}$ $$(1+2i)+(1-2i)-3 = -a$$ $$a = 1$$	**2 marks** – Correct solution **1 mark** – Substantially correct

	Question 11 continued…	
(d)(i) *MEX12-6*	Resistance $F = kv$ $40 = k \times 10$ $k = 4$ Resultant force $F = ma = -kv$ $ma = -4v$ $a = -\frac{4}{m}v$	**1 mark** – Correct solution also accept $a = -4v$ note: resistance works *against* motion, so as $v > 0$, $a < 0$. No marks awarded if acceleration was not shown to be –ve.
(d)(ii) *MEX12-6*	$\frac{dv}{dt} = g - kv$ $(m = 1)$ $= 10 - 4v$ $\frac{dt}{dv} = \frac{1}{10 - 4v}$ $\int_0^t dt = \int_0^v \frac{1}{10 - 4v} dv$ $\left[t\right]_0^t = \left[-\frac{1}{4}\ln\lvert 10 - 4v\rvert\right]_0^v$ $t = -\frac{1}{4}\left[\ln\lvert 10 - 4v\rvert - \ln 10\right]$ $t = \frac{1}{4}\ln\left\lvert\frac{10}{10 - 4v}\right\rvert$ $e^{4t} = \frac{10}{10 - 4v}$ $10 - 4v = \frac{10}{e^{4t}}$ $-4v = \frac{10}{e^{4t}} - 10$ $v = \frac{5}{2} - \frac{5}{2e^{4t}}$	**3 marks** – Correct solution **2 marks** – Substantially correct solution **1 mark** – Partial progress towards correct solution Note: generic, or rote-learned solution (in terms of g and k) awarded ***maximum*** 2 marks. *Values* from question were required in final answer

Outcomes Addressed in this Question

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts.
MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings.
MEX12-3 uses vectors to model and solve problems in two and three dimensions
MEX12-6 uses mechanics to model and solve practical problems

Outcome	Solutions	Marking Guidelines
(a) **MEX12-3**	(a) (i) $\overrightarrow{AB}=\overrightarrow{BD}=\begin{pmatrix}2\\-5\\7\end{pmatrix}$ $\quad \overrightarrow{AD}=2\overrightarrow{AB}=\begin{pmatrix}4\\-10\\14\end{pmatrix}$ Hence position vector for *D*: $\begin{pmatrix}2+4\\3-10\\-4+10\end{pmatrix}=\begin{pmatrix}6\\-7\\10\end{pmatrix}$	**(a) (i) 2 marks:** Correct solution 1 mark: Significant progress.
	(ii) $\overrightarrow{AC}=\begin{pmatrix}a-2\\2\\2\end{pmatrix}$ $\left\lvert\overrightarrow{AC}\right\rvert=4 \qquad \therefore (a-2)^2+2^2+2^2=16$ $a=2-2\sqrt{2}$ since *a* is negative.	**(ii) 3 marks:** Correct solution, including all aspects shown. **2 marks:** One aspect of solution not shown. **1 mark**: Relevant progress.
(b) **MEX12-1** **MEX12-2**	(b) Assume *p, q, r* are integers. Hence the Pythagorean Triads: $p^2+q^2=r^2$ and $(p+1)^2+(q+1)^2=(r+1)^2$ $\therefore p^2+2p+1+q^2+2q+1=r^2+2r+1$ $r^2+2p+2q+2=r^2+2r+1$ $2(p+q)=2r-1$ LHS even; RHS odd Gives a contradiction, so the assumption that *p*, *q*, and *r* are integers must be false.	**(b) 3 marks:** Full solution with reasoning and explanations **2 marks:** One aspect of solution not shown. **1 mark**: Relevant progress.

(c) **MEX12-6**	(c) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x+3$ $\frac{1}{2}v^2=\int x+3\,dx$ $=\frac{1}{2}x^2+3x+c \qquad x=0 \text{ gives } v=3$ $\frac{1}{2}\left(3^2\right)=c=\frac{9}{2}$ $\therefore \frac{1}{2}v^2=\frac{x^2}{2}+3x+\frac{9}{2}$ $v^2=\pm\left(x+3\right)^2$ Initial conditions give positive velocity towards the positive side of the *x*-axis, so *v* can only ever be positive. Hence $v=x+3$	**(c) (i) 2 marks:** Correct solution including justification for positive sqrt only. **1 mark:** Significant progress.
	(c)(ii) $\frac{dx}{dt}=x+3 \to \frac{dt}{dx}=\frac{1}{x+3}$ $t=\ln\left(x+3\right)+c \qquad t=0 \text{ gives } x=0$ $0=\ln 3+c \quad \to \quad c=-\ln 3$ $\therefore t=\ln\left(\frac{x+3}{3}\right)$ $e^t=\frac{x+3}{3}$ $x=3e^t-3$	**(ii) 2 marks:** Correct solution including all relevant steps. **1 mark:** Significant progress towards correct solution.
(d) **MEX 12-3**	(d)(i) $x^2+2x+1+y^2-5y+\frac{25}{4}+z^2-4z+4=\frac{45}{4}-k$ $\left(x+1\right)^2+\left(y-\frac{5}{2}\right)^2+\left(z-2\right)^2=\frac{45-4k}{4}$ $\therefore$ Centre = $\left(-1,\frac{5}{2},2\right)$ Radius $=\frac{\sqrt{45-4k}}{2}$	**d)(i) 2 mark:** Both components answered correctly. **1 mark:** At least one component correct.
	(d) (ii) Distance from the *x*-*z* plane is the *y*-value. $\frac{\sqrt{45-4k}}{2}=\frac{5}{2}$ $45-4k=25 \quad \to \quad k=5$	**(ii)** **1 mark:** Correct solution (CFPA (i)).

Year 12	Mathematics Extension 2 2022	TASK 4 (TRIAL)
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-5 applies techniques of integration to structured and unstructured problems

Part / Outcome	Solutions	Marking Guidelines
(a)(i) *MEX12-1*	If x is even, then x^2-6x+5 is odd	**1 mark** – Correct solution
(a)(ii) *MEX12-1*	Suppose that x is even, ie $x = 2k$ for integer k. $x^2-6x+5=(2k)^2-6(2k)+5$ $=4k^2-12k+5$ $=2(2k^2-6k+2)+1$ $=2m+1$, where m is an integer Hence, x^2-6x+5 is odd and the statement is proven by proving the contrapositive.	**2 marks** – Correct solution **1 mark** – Substantially correct
(b) *MEX12-5*	$t=\tan\frac{x}{2}$ $\frac{x}{2}=\tan^{-1}t$ $x=2\tan^{-1}t$ $dx=\frac{2}{1+t^2}dt$ $\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{1+\sin x+\cos x}dx=\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{1+\frac{2t}{1+t^2}+\frac{1-t}{1+t^2}}\frac{2}{1+t^2}dt$ $=\int_0^{\frac{1}{\sqrt{3}}}\frac{4dt}{1+t^2+2t+1-t^2}$ $=\int_0^{\frac{1}{\sqrt{3}}}\frac{2dt}{t+1}=2\left[\ln(1+t)\right]_0^{\frac{1}{\sqrt{3}}}$ $=2\ln\left(1+\frac{1}{\sqrt{3}}\right)$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution

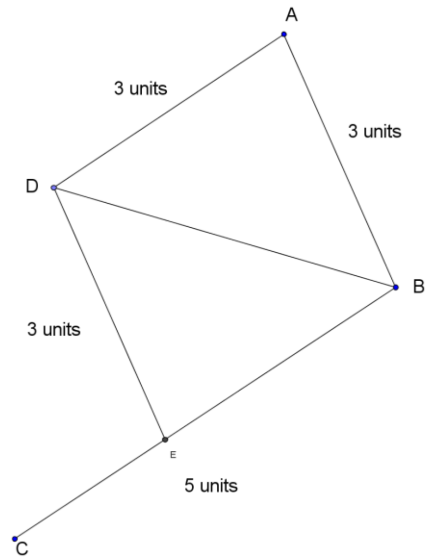
	Question 13 continued…	
(c)(i) *MEX12-5*	$u = \ln^n x \qquad v' = x^m$ $u' = \dfrac{n\ln^{n-1} x}{x} \qquad v = \dfrac{x^{m+1}}{m+1} dx$ $\int uv'\,dx = uv - \int u'v\,dx$ $\int x^m \ln^n x\,dx = \dfrac{x^{m+1}}{m+1}\ln^n x - \int \dfrac{x^{m+1}}{m+1}\dfrac{n\ln^{n-1} x}{x}dx \quad *$ $= \dfrac{x^{m+1}}{m+1}\ln^n x - \dfrac{n}{m+1}\int x^m \ln^{n-1} x\,dx$ $= \dfrac{x^{m+1}}{m+1}\ln^n x - \dfrac{n}{m+1} I_{n-1}$	**2 marks** – Correct solution **1 mark** – Substantially correct NB: this is a *show* question, and as such, it should be explicitly shown where the terms in this expression $(*)$ come from
(c)(ii) *MEX12-5*	$\int_1^2 x^3 \ln^4 x\,dx$ $= \left[\dfrac{x^4}{4}\ln^4 x\right]_1^2 - I_3$ $= \left[\dfrac{2^4}{4}\ln^4 2 - 0\right] - \left\{\left[\dfrac{x^4}{4}\ln^3 x\right]_1^2 - \dfrac{3}{4}I_2\right\}$ $= 4\ln^4 2 - \left[\dfrac{2^4}{4}\ln^3 2 - 0\right] + \dfrac{3}{4}\left\{\left[\dfrac{x^4}{4}\ln^2 x\right]_1^2 - \dfrac{2}{4}I_1\right\}$ $= 4\ln^4 2 - 4\ln^3 2 + \dfrac{3}{4}\left[\dfrac{2^4}{4}\ln^2 2 - 0\right] - \dfrac{3}{4}\times\dfrac{2}{4}\left\{\left[\dfrac{x^4}{4}\ln x\right]_1^2 - \dfrac{1}{4}I_0\right\}$ $= 4\ln^4 2 - 4\ln^3 2 + 3\ln^2 2 - \dfrac{3}{4}\times\dfrac{2}{4}\left[\dfrac{2^4}{4}\ln 2 - 0\right] + \dfrac{3}{4}\times\dfrac{2}{4}\times\dfrac{1}{4}\int_1^2 x^3\,dx$ $= 4\ln^4 2 - 4\ln^3 2 + 3\ln^2 2 - \dfrac{3}{2}\ln 2 + \dfrac{3}{32}\left[\dfrac{x^4}{4}\right]_1^2$ $= 4\ln^4 2 - 4\ln^3 2 + 3\ln^2 2 - \dfrac{3}{2}\ln 2 + \dfrac{3}{32}\left[4 - \dfrac{1}{4}\right]$ $= 4\ln^4 2 - 4\ln^3 2 + 3\ln^2 2 - \dfrac{3}{2}\ln 2 + \dfrac{45}{128}$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution

	Question 13 continued…	
(d)(i) *MEX12-4*	$\arg i = \frac{\pi}{2}$ and $\lvert i \rvert = 1$, so $z = 1e^{i\frac{\pi}{2}}$	**1 mark** – Correct solution
(d)(ii) *MEX12-4*	$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}}$ $= e^{-\frac{\pi}{2}} \approx 0.208$	**2 marks** – Correct solution **1 mark** – Substantially correct solution
(d)(iii) *MEX12-4*	$\arg i = \frac{-3\pi}{2} \rightarrow i = e^{-i\frac{3\pi}{2}}$ $i^i = \left(e^{-i\frac{3\pi}{2}}\right)^i = e^{\frac{3\pi}{2}} \approx 111.318$ Alternate way to handle (c)(ii)… $I_0 = \int_1^2 x^3 (\ln x)^0 \, dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{15}{4}$ $I_1 = \left[\frac{x^4}{4}\ln x\right]_1^2 - \frac{1}{4} I_0$ $= 4\ln x - \frac{1}{4}\left(\frac{15}{4}\right) = 4\ln x - \frac{15}{16}$ $I_2 = \left[\frac{x^4}{4}\ln^2 x\right]_1^2 - \frac{2}{4} I_1$ $= 4\ln^2 x - \frac{2}{4}\left[4\ln x - \frac{15}{16}\right]$ $= 4\ln^2 x - 2\ln x + \frac{15}{32}$ $I_3 = \left[\frac{x^4}{4}\ln^3 x\right]_1^2 - \frac{3}{4} I_2$ $= 4\ln^3 x - \frac{3}{4}\left[4\ln^2 x - 2\ln x + \frac{15}{32}\right]$ $= 4\ln^3 x - 3\ln^2 x + \frac{3}{2}\ln x - \frac{45}{128}$ $I_4 = \left[\frac{x^4}{4}\ln^4 x\right]_1^2 - \frac{4}{4} I_3$ $= 4\ln^4 x - \frac{4}{4}\left[4\ln^3 x - 3\ln^2 x + \frac{3}{2}\ln x - \frac{45}{128}\right]$ $= 4\ln^4 x - 4\ln^3 x + 3\ln^2 x - \frac{3}{2}\ln x + \frac{45}{128}$	**1 mark** – Correct solution

Outcomes Addressed in this Question

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts.
MEX12-3 uses vectors to model and solve problems in two and three dimensions
MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems.
MEX12-6 uses mechanics to model and solve practical problems

Outcome	Solutions	Marking Guidelines
(a) **MEX12-6**	(a) (i) $ma = mg - kmv^2$ where $k > 0$ $a = g - kv^2$ $v\dfrac{dv}{dx} = g - kv^2$ $\dfrac{dx}{dv} = \dfrac{v}{g-kv^2}$ $x = -\dfrac{1}{2k}\ln\left(g-kv^2\right)+c$ $x=0$ when $v=0$ $\therefore c = \dfrac{1}{2k}\ln g$ $x = \dfrac{1}{2k}\ln g - \dfrac{1}{2k}\ln\left(g-kv^2\right)$ $x = \dfrac{1}{2k}\ln\left(\dfrac{g}{g-kv^2}\right)$ $\therefore e^{2kx} = \dfrac{g}{g-kv^2}$ $g - kv^2 = ge^{-2kx}$ $v^2 = \dfrac{g}{k}\left(1-e^{-2kx}\right)$	**(a) (i) 3 marks:** Correct solution, including all aspects shown. **2 marks:** One aspect of solution not shown. **1 mark**: Relevant progress.
	(a) (ii) As $x\to\infty, e^{-2kx}\to 0$ $\therefore v^2 \to \dfrac{g}{k}$ $v_T = \sqrt{\dfrac{g}{k}}$	**(ii) 1 mark:** Correct answer with justification.
	(a)(iii) Substituting for *x*. $v^2 = \left(1-e^{-2k\left(\frac{1}{2k}\ln\frac{4}{3}\right)}\right)\dfrac{g}{k}$ $= \left(1-e^{\frac{3}{4}}\right)\dfrac{g}{k}$ $= \left(1-\dfrac{3}{4}\right)\dfrac{g}{k}$ $\therefore v = \dfrac{1}{2}\sqrt{\dfrac{g}{k}} = \dfrac{1}{2}v_T$	**(iii) 2 marks:** Correct substitution and reasoning. **1 mark:** Substantial progress.

(b) **MEX12-6**	(b)(i) $\frac{1}{2}v^2 = 2\left(2x - x^2\right)$ $\frac{d}{dx}\frac{1}{2}v^2 = 2\left(2 - 2x\right)$ $= -4\left(x - 1\right)$ (b)(ii) $\ddot{x}$ is in the form $-n^2\left(x - c\right)$ Therefore motion is SHM. (b)(iii) $\ddot{x} = 0$ when $x = 1$ $\therefore v^2 = 4\left(2(1) - (1)^2\right) = 4$ $\therefore$ Max speed $= 2ms^{-1}$	**(b) (i)2 marks:** Correct substitution and reasoning. **1 mark:** Substantial progress. **(ii) 1 mark:** Correct answer referring to how this example contains the relevant features. **(iii) 1 mark:** Correct calculation.
(c) **MEX12-3**	(c) $\left\|\overrightarrow{BA}\right\| = 3 \qquad \left\|\overrightarrow{BC}\right\| = 5$ Consider the diagram below that utilises the property of a rhombus, where the diagonal bisects the angles it passes through.  For *BD* to bisect $\angle ABC$ we have the following vector sum. $\overrightarrow{BD} = \overrightarrow{BA} + \frac{3}{5}\overrightarrow{BC}$ $= \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ This is a direction vector for line *L*. Line L also passes through point *B*. $L = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	**(c) 2 marks:** Correct derivation of gradient vector and inserting it into the correct vector equation of a straight line. **1 mark:** Some relevant progress.

(d) **MEX12-1** **MEX12-4**	(d) $w^4 = (w-2)^4 \quad \rightarrow \qquad 0 = w^3 - 3w^2 + 4w - 2$ Factors are $(w-1)(w^2 - 2w + 2) = 0$ $w = 1, 1 \pm i$ *NOTE:* There were many methods available to solve this question. The one above was the most common but not the most concise. The most concise method involved solving $\left(\frac{w}{w-2}\right)^4 = 1$ and utilising the properties of the Argand Diagram.	**(d) 3 marks:** Correct solution. **2 marks:** Solution almost fully correct. **1 mark:** Relevant progress.

Year 12	Mathematics Extension 2 2022	TASK 4 (TRIAL)
Question No. 15	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

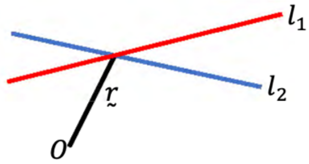
MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Part / Outcome	Solutions	Marking Guidelines

(a)(i) *MEX12-3*

Solving l_1 and l_2 simultaneously,



$$\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

so, $9 + \lambda = 2 + 2\mu$

$\lambda = 2\mu - 7 \quad \ldots\boxed{1}$

and $13 + 4\lambda = -1 + \mu$

$\mu = 4\lambda + 14 \quad \ldots\boxed{2}$

sub $\boxed{1} \rightarrow \boxed{2}$

$\mu = 4(2\mu - 7) + 14$

$-7\mu = -14$

$\mu = 2$

and $\lambda = 2(2) - 7 = -3$

$$\therefore \underset{\sim}{r} = 9\underset{\sim}{i} + 13\underset{\sim}{j} - 3\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$
$$= 9\underset{\sim}{i} + 13\underset{\sim}{j} - 3\underset{\sim}{k} - 3\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$
$$= 6\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$$

3 marks – Correct solution

2 marks – Substantially correct

1 mark – Partial progress towards correct solution

(a)(ii) *MEX12-3*

l_1 has direction vector $\underset{\sim}{a} = \underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

l_2 has direction vector $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\left|\underset{\sim}{a}\right| \times \left|\underset{\sim}{b}\right|}$$

$$= \frac{1 \times 2 + 4 \times 1 + (-2) \times 1}{\sqrt{1^2 + 4^2 + (-2)^2} \times \sqrt{2^2 + 1^2 + 1^2}} = \frac{4}{\sqrt{21} \times \sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{126}}\right)$$
$$\approx 69.1°$$

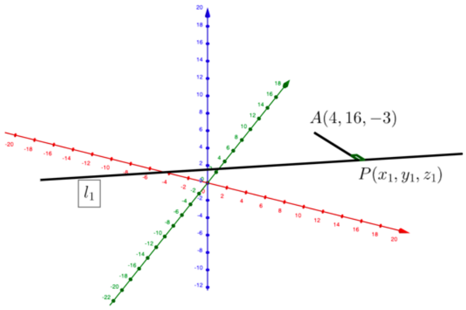
2 marks – Correct solution

1 mark – Substantially correct

Question 15 continued…

(a)(iii)
MEX12-3

P lies on l_1, so



$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

ie $x_1 = 9+\lambda$

$y_1 = 13+4\lambda$ $\quad *$

$z_1 = -3-2\lambda$

Now, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$$= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix} = \begin{pmatrix} 9+\lambda-4 \\ 13+4\lambda-16 \\ -3-2\lambda+3 \end{pmatrix} = \begin{pmatrix} 5+\lambda \\ -3+4\lambda \\ -2\lambda \end{pmatrix}$$

noting that l_1 has direction vector $\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$,

$$l_1 \perp \overrightarrow{AP}$$

then, $\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)\cdot \overrightarrow{AP} = 0$

$$1(5+\lambda)+4(4\lambda-3)-2(-2\lambda)=0$$

$$-7+21\lambda=0$$

$$\lambda = \frac{1}{3}$$

Hence, P is $\left(9+\tfrac{1}{3},\ 13+4\left(\tfrac{1}{3}\right),\ -3-2\left(\tfrac{1}{3}\right)\right)$ from $*$

$$=\left(\frac{28}{3}, \frac{43}{3}, -\frac{11}{3}\right)$$

3 marks – Correct solution

2 marks – Substantially correct

1 mark – Partial progress towards correct solution

(b)(i)
MEX12-4

By de Moivre's Theorem and Binomial Theorem:

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$$

Equating real and Imaginary parts:

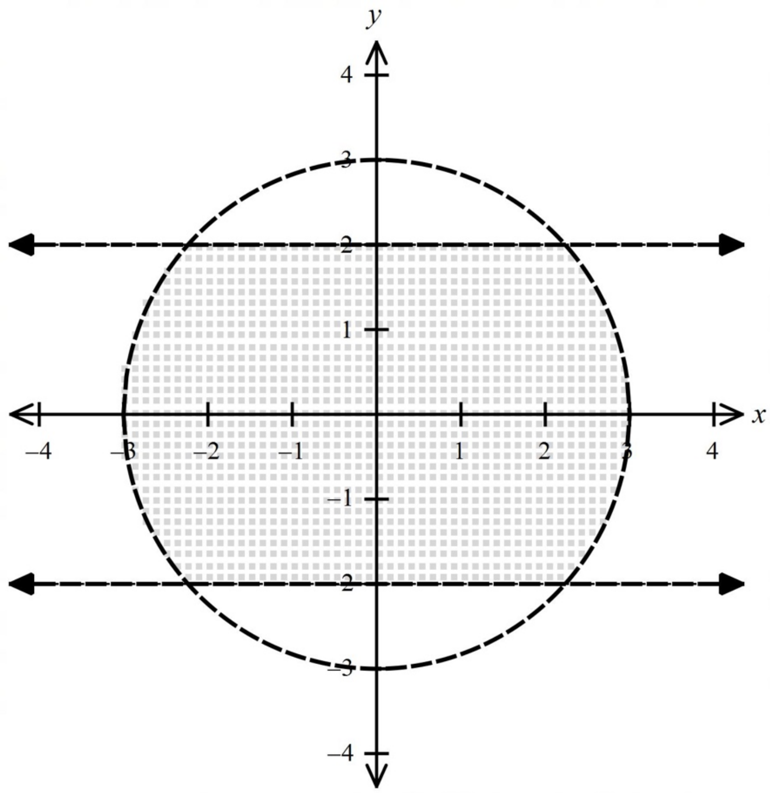
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

3 marks – Correct solution

2 marks – Substantially correct

1 mark – Partial progress towards correct solution
Correctly uses de Moivre's Theorem or equivalent merit

	$\tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$ $= \dfrac{\dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta}}{\dfrac{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}{\cos^5\theta}}$ $*$ *show* $= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^5\theta}$	Note, it is important to give *some* indication as to how this expression ended up at the **given** answer. Dividing through by $\cos^5\theta$ needed to be shown, or at least stated.
(b)(ii) *MEX12-4*	if $\tan 5\theta = 0$ then $5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0$ $\tan\theta\left(5 - 10\tan^2\theta + \tan^3\theta\right) = 0$ Also, $\tan 5\theta = 0 \rightarrow \theta = \dfrac{\pi}{5}, \dfrac{2\pi}{5}$ and, as $\tan\dfrac{\pi}{5} \neq 0$, $\tan\dfrac{2\pi}{5} \neq 0$ then $\theta = \dfrac{\pi}{5}$, $\theta = \dfrac{2\pi}{5}$ are solutions of $5 - 10\tan^2\theta + \tan^4\theta = 0$ Hence $\tan\dfrac{\pi}{5}, \tan\dfrac{2\pi}{5}$ are solutions of $5 - 10x^2 + x^4 = 0$ and $\tan^2\dfrac{\pi}{5}, \tan^2\dfrac{2\pi}{5}$ are solutions of $5 - 10x + x^2 = 0$	**2 marks** – Correct solution **1 mark** – Substantially correct solution **0 marks** – Using a calculator
(c) *MEX12-4*		**2 marks** – Correct solution **1 mark** – Substantially correct solution

Outcomes Addressed in this Question

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts.
MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings.
MEX12-5 applies techniques of integration to structured and unstructured problems

Outcome	Solutions	Marking Guidelines
(a) **MEX12-5**	(a) Substitution method:	**(a)3 marks:** Substituting into a new integral; altering the form of the new integrand, solving and expressing answer in terms of *x*. **2 marks:** 2 components correct. **1 mark**: Correct substitution step..

$$u^2 = x \quad \rightarrow \qquad 2udu = dx$$

$$\int \frac{\sqrt{x}}{4+x}dx = \int \frac{u}{4+u^2}.2udu$$

$$= 2\int \frac{4+u^2-4}{4+u^2}du$$

$$= 2\int du - 4\int \frac{\frac{1}{2}}{1+\left(\frac{u}{2}\right)^2}du$$

$$= 2u - 4\tan^{-1}\left(\frac{u}{2}\right) + c$$

$$= 2\sqrt{x} - 4\tan^{-1}\left(\frac{\sqrt{x}}{2}\right) + c$$

Outcome	Solutions	Marking Guidelines
(b) **MEX12-1** **MEX12-2**	(b)	**(b)3 marks:** All aspects of induction correct. **2 marks:** Significant progress. **1 mark**: Relevant progress.

RTP: $T_n = 3\times 2^n + 1$

Step 1: Prove true for *n*=1.

T_1=7 $\quad 3\times 2^1 + 1 = 7 \quad$ Hence true for *n* = 1.

Step 2: Assume: $T_k = 3\times 2^k + 1$

Prove: $T_{k+1} = 3\times 2^{k+1} + 1$

$$LHS = 2\left(T_k\right) - 1$$

$$= 2\left(3\times 2^k + 1\right) - 1$$

$$= 3\times 2^{k+1} + 2 - 1$$

$$= 3\times 2^{k+1} + 1 \qquad = RHS$$

Thus proven by mathematical induction.

Outcome	Solutions	Marking Guidelines
(c) **MEX12-1**	(c)(i)	**(c) (i) 2 marks:** Correct expansion and use of the axiom given. **1 mark:** Substantial progress.

$$\left(a+b+c\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1$$

$$= 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$\geq 3+2+2+2 = 9$$

$$\therefore LHS \geq RHS$$

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c(ii)
From (i), let $a \to a+b \quad b \to b+c \qquad c \to a+c$
$$\left((a+b)+(b+c)+(a+c)\right)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}\right) \ge 9$$
$$1+\frac{a+b}{b+c}+\frac{a+b}{a+c}+\frac{b+c}{a+b}+1+\frac{b+c}{a+c}+\frac{a+c}{a+b}+\frac{a+c}{b+c}+1 \ge 9$$
$$\frac{a+b+2c}{a+b}+\frac{2a+b+c}{b+c}+\frac{a+2b+c}{a+c} \ge 6$$
$$1+\frac{2c}{a+b}+1+\frac{2a}{b+c}+1+\frac{2b}{a+c} \ge 6$$
$$\to \frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \ge \frac{3}{2}$$
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(ii) 3 marks: Correct solution using hence from (i) with justification.
2 marks: Significant progress towards correct solution.
1 mark: Relevant progress.
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(d)
MEX 12-2
MEX12-5
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(d)
Substitution: Let $u = \frac{1}{x} \quad \to \qquad dx = \frac{-1}{u^2}du$
$x = 1$ gives $u = 1 \qquad x = 0$ gives $u \to \infty$
$$\int_0^1 \frac{x^3+1}{x^5+1}dx = \int_\infty^1 \frac{\frac{1}{u^3}+1}{\frac{1}{u^5}+1}\left(\frac{-du}{u^2}\right)$$
$$= \int_1^\infty \frac{\frac{1}{u^3}+1}{\frac{1}{u^3}+u^2}du$$
$$= \int_1^\infty \frac{1+u^3}{1+u^5}du$$
$$\therefore \int_0^1 \frac{x^3+1}{x^5+1}dx = \int_1^\infty \frac{1+x^3}{1+x^5}dx$$
And so the area from x=0 to x=1 is half of the area for $x \ge 0$.
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(d)4 marks: Correct new integral from substitution given; correct new integral limits; simplification; conclusion.
3 marks: All aspects shown with one feature unsubstantiated.
2 marks: Significant progress.
1 mark: Some relevant progress.
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